

Spatial pattern formation in external noise: theory and simulation

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Spatial pattern formation in excitable fluctuating media was researched analytically from the point of view of the order parameters concept. The reaction-diffusion system in external noise is considered as a model of such medium. Stochastic equations for the unstable mode amplitudes (order parameters), dispersion equations for the unstable mode averaged amplitudes, and the Fokker-Planck equation for the order parameters have been obtained. The developed theory makes it possible to analyze different noise-induced effects, including the variation of boundaries of ordering and disordering phase transitions depending on the parameters of external noise

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I. INTRODUCTION

Noise is present in real systems of any type. The influence of external noise on nonlinear open distributed systems is very diverse and sometimes leads to quite unexpected results. It is known, that noise-induced spatial patterns [1–7] and fronts [8, 9], noise-induced resonant pattern and frequency-locking phenomena [10], pure noise-induced phase transitions [11–14], noise-induced phase separation [15], noise-induced spatiotemporal intermittency [16], spatiotemporal stochastic resonance [17–19], noise-supported traveling and noise-sustained convective structures [20–22], noise-induced synchronization [23–25], etc., may arise in such systems.

Theoretical study of spatio-temporal dynamics of nonlinear open distributed systems is carried out using different methods [26–34].

Stability of the homogeneous state with respect to small perturbations can be analyzed in linear approximation. However, linear approximation is not acceptable to describe the evolution of the system near the threshold of self-organization, as the phenomenon self-organization itself is essentially a nonlinear effect.

There is an approach to the study of noise-induced phenomena, based on the well-known mean-field approximation [4, 12, 15, 26, 27]. In this approximation it is assumed, that the interaction between a certain spatial point and its nearest neighbors occurs through the field, whose value corresponds to the statistically average field at this point. Herewith, a suitable way is used to carry out the discretization of the space of the initial distributed system and the Fokker-Planck equation (FPE) for the multivariate probability density function can

be written for field values in the points received by a regular lattice. The obtained FPE is integrated over the values of the field at all points except the given one. This leads to FPE for the one-dimensional probability density function values of the field at a given point. In the latter equation the conditional average values of the field at neighboring points are replaced by an average value of the field at a given point. This approach can predict the existence of noise-induced “disorder-order-disorder” phase transitions.

Another analytical approach to the study of nonequilibrium phase transitions with the spatial pattern formation is based on generalized Ginzburg-Landau equations for the order parameters of the systems. This approach makes it possible to describe the behavior of the system near the transition point. It is based on the separation of eigenmodes of the system into damped (stable) and undamped (unstable) modes (order parameters), and into adiabatic elimination of stable modes. The method of generalized Ginzburg-Landau equations for the systems, the right-hand side of which contains additive white noise, is described in [29].

In papers [14, 34] spatio-temporal evolution of the nonequilibrium extended systems is investigated by means of dynamic renormalization groups. In [14] it is shown that under certain conditions a new genuine nonequilibrium universality class arises due to the presence of multiplicative noise.

Moreover, there are approaches based on analysis of correlation functions of the dynamic variables of the system or structure functions [3, 12], on the study of higher-order moments [6, 32], and others [30, 31, 33, 36].

The aim of this paper is to develop a theory, which would allow us from the unified point of view of the concept of order parameters, to carry out a consistent and detailed study of spatial pattern formation spontaneously arising in open nonlinear distributed systems with external noise both in the vicinity of the transition point and away from it.

II. STOCHASTIC EQUATIONS FOR THE ORDER PARAMETERS

Systems of the reaction-diffusion type

$$\frac{\partial x_k}{\partial t} = P_k(x_1, x_2, x_3, \dots, \boldsymbol{\eta}, \mathbf{r}, t) + D_k \nabla^2 x_k, k = 1, 2, 3 \dots \quad (1)$$

are one of the basic models of a nonequilibrium excitable medium. In equation (1) x_k are the medium state functions (dynamic variables), $P_k(x_1, x_2, x_3, \dots, \boldsymbol{\eta}, \mathbf{r}, t)$ are nonlinear functions, that define the interaction and the evolution of the component x_k in space and in time, D_k are diffusion coefficients of components, $\boldsymbol{\eta} = (\chi_1, \dots, \chi_n, \eta_1, \dots, \eta_s)$ is the vector, whose components are the control parameters describing the effect of the external environment on the system. Further, without loss of generality, we consider a two-component system of type (1). Nevertheless, the proposed research method is easily extended to multicomponent systems.

In external noise the spatio-temporal dynamics of nonequilibrium systems (1) for $k = 2$ can be described by the following system of equations:

$$\begin{aligned} \frac{\partial x_1}{\partial t} &= P_1(x_1, x_2, \chi_{10}, \dots, \chi_{m0}, \dots, \chi_n) + \sum_{j=1}^m f_{1j}(\mathbf{r}, t) P_{1j}(x_1, x_2, \chi_{m+1}, \dots, \chi_{n'}) + D_1 \nabla^2 x_1, \\ \frac{\partial x_2}{\partial t} &= P_2(x_1, x_2, \eta_{10}, \dots, \eta_{l0}, \dots, \eta_s) + \sum_{j=1}^l f_{2j}(\mathbf{r}, t) P_{2j}(x_1, x_2, \eta_{l+1}, \dots, \eta_{s'}) + D_2 \nabla^2 x_2, \end{aligned} \quad (2)$$

where m and l are the number of fluctuating parameters in the first and second equations, respectively, χ_{j0} , η_{j0} are spatio-temporal average parameters, $f_{ij}(\mathbf{r}, t)$ ($i=1,2$) are random fields describing the noise of the appropriate parameters with respect to their mean values with $\langle f_{ij}(\mathbf{r}, t) \rangle = 0$.

We define the statistical properties of random fields $f_{ij}(\mathbf{r}, t)$ according to the properties of the environment. Fluctuations in the environment represent the summarized effect of many weakly coupled factors. It follows from the central limit theorem, that fluctuations of the external source have a Gaussian distribution. The ergodic Markovian and Gaussian properties of fluctuating environment limits the choice of random fields for modeling the fluctuations of the environment by a stationary homogeneous isotropic Gaussian field with the exponential time-correlation function [28]:

$$K[f_{ij}(\mathbf{r}, t), f_{i'j'}(\mathbf{r}', t')] = \Phi_i(|\mathbf{r} - \mathbf{r}'|) \exp(-k_{ti}|t - t'|) \delta_{ii'} \delta_{jj'}, \quad (3)$$

where $\Phi_i(|\mathbf{r} - \mathbf{r}'|)$ define the spatial dependence of correlation functions of the random fields. The cross-correlation of the fields $f_{ij}(\mathbf{r}, t)$ and $f_{i'j'}(\mathbf{r}, t)$ is absent. The correlation time $\tau_t = k_t^{-1}$ is the characteristic time scale of external noise. Hereafter, we use the notation $K[F_1, F_2]$, that is defined by the equality $K[F_1, F_2] = \langle F_1 F_2 \rangle - \langle F_1 \rangle \langle F_2 \rangle$ for the correlation function.

Let us assume for simplicity, that $m = l = 1$, and introduce the dimensionless variables $\tau = \chi_{10} t$ and $\mathbf{r}' = \mathbf{r} \sqrt{\chi_{10}/D_1}$, where χ_{10} is a parameter, that has the dimension of inverse time. Now the system (2) can be rewritten as

$$\begin{aligned} \frac{\partial x_1}{\partial \tau} &= P'_1(x_1, x_2, \chi_{10}, \dots, \chi_n) + f_{11}(\mathbf{r}', \tau) P'_{11}(x_1, x_2, \chi_2, \dots, \chi_{n'}) + \nabla'^2 x_1, \\ \frac{\partial x_2}{\partial \tau} &= P'_2(x_1, x_2, \eta_{10}, \dots, \eta_s) + f_{21}(\mathbf{r}', \tau) P'_{21}(x_1, x_2, \eta_2, \dots, \eta_{s'}) + D \nabla'^2 x_2. \end{aligned} \quad (4)$$

Functions with primes are different from the corresponding functions without primes by a multiplier χ_{10}^{-1} . $D = D_2/D_1$. Hereafter, the primes are dropped for simplicity.

Suppose, that in the deterministic case the values of parameter $\chi_1, \dots, \chi_n, \eta_1, \dots, \eta_s$ are such, that there are stable stationary states x_{10} and x_{20} defined by the equations $P_1(x_1, x_2, \chi_1, \dots, \chi_n) = 0$ and $P_2(x_1, x_2, \eta_1, \dots, \eta_s) = 0$.

We write the equations (4) in the operator form. Simultaneously, we select from its right-hand side linear $K(\nabla^2)\mathbf{q}$, nonlinear $\mathbf{g}(\mathbf{q})$, and random $\mathbf{F}(\mathbf{r}, \tau)$ components [29]

$$\frac{\partial \mathbf{q}}{\partial \tau} - K(\nabla^2)\mathbf{q} = \mathbf{g}(\mathbf{q}) + \mathbf{F}(\mathbf{r}, \tau). \quad (5)$$

Vector \mathbf{q} describes the deviation of the dynamic variables from their equilibrium values: $\mathbf{q} = (x_1 - x_{10}, x_2 - x_{20})$. A linear operator $K(\nabla^2)$ takes the form

$$K(\nabla^2) = \begin{pmatrix} a_{11} + \nabla^2 & a_{12} \\ a_{21} & a_{22} + D\nabla^2 \end{pmatrix}, \quad a_{ij} = \left. \frac{\partial P_i}{\partial x_j} \right|_{x_{10}, x_{20}}, \quad i, j = 1, 2. \quad (6)$$

Vector $\mathbf{g}(\mathbf{q})$ contains quadratic and cubic nonlinearities obtained by series expansion of the right-hand deterministic side of the equation (4). Its components are defined as follows

$$g_i(\mathbf{q}) = \sum_{\mu, \nu=1}^2 g_{i, \mu\nu}^{(2)} q_\mu q_\nu + \sum_{\mu, \nu, \kappa=1}^2 g_{i, \mu\nu\kappa}^{(3)} q_\mu q_\nu q_\kappa, \quad (7)$$

where $g_{i, \mu\nu}^{(2)} = \frac{1}{2!} \left. \frac{\partial^2 P_i}{\partial x_\mu \partial x_\nu} \right|_{x_{10}, x_{20}}$, $g_{i, \mu\nu\kappa}^{(3)} = \frac{1}{3!} \left. \frac{\partial^3 P_i}{\partial x_\mu \partial x_\nu \partial x_\kappa} \right|_{x_{10}, x_{20}}$.

Vector \mathbf{F} contains the random fields:

$$\mathbf{F} = \begin{pmatrix} f_{11}(\mathbf{r}, \tau) P_{11}(x_1, x_2, \chi_2, \dots, \chi_{n'}) \\ f_{21}(\mathbf{r}, \tau) P_{21}(x_1, x_2, \eta_2, \dots, \eta_{s'}) \end{pmatrix}.$$

To research the stability of the stationary state of a deterministic system, we assume that the vector \mathbf{q} has the form $\mathbf{q} = \mathbf{q}_0 \exp(\lambda\tau + i\mathbf{k}\mathbf{r})$. The respective characteristic equation $\lambda^2 - \alpha\lambda + \beta = 0$ has solutions

$$\lambda_{1,2}(\mathbf{k}) = \frac{\alpha(k) \pm \sqrt{\alpha^2(k) - 4\beta(k)}}{2}, \quad (8)$$

where $\alpha(k) = \text{Tr}(a_{ij}) - (1 + D)k^2$, $\beta(k) = \text{Det}(a_{ij}) - (Da_{11} + a_{22})k^2 + Dk^4$.

Conditions $\alpha(k) < 0$ and $\beta(k) \leq 0$ define aperiodic instability, herewith one positive real root of eq. (8) appears: $\text{Re}(\lambda_1) \geq 0$, $\text{Im}(\lambda_{1,2}) = 0$. Further we shall consider only this case.

Suppose, that one of the parameters $\chi_1, \dots, \chi_n, \eta_1, \dots, \eta_s$, for example χ_2 , is bifurcation, i.e., at some critical value of this parameter χ_{2c} there exists a critical wave number k_c , when the conditions $\text{Re}(\lambda_1(k_c)) = 0$, $\left. \frac{d(\text{Re}\lambda_1(k))}{dk} \right|_{k=k_c} = 0$ are fulfilled.

Represent the vector \mathbf{q} in the form of superposition

$$\mathbf{q}(\mathbf{r}, \tau) = \sum_{\mathbf{k}', j} \mathbf{O}^{(j)}(\mathbf{k}') \xi_{\mathbf{k}'}^{(j)}(\tau) e^{i\mathbf{k}'\mathbf{r}}, \quad (9)$$

where $\mathbf{O}^{(j)}(\mathbf{k})$ are eigenvectors of operator $K(\nabla^2)$, $\xi_{\mathbf{k}}^{(j)}(\tau)$ are unknown amplitudes, $\exp(i\mathbf{k}\mathbf{r})$ are eigenfunctions of operator ∇^2 . Here it is assumed, that the vector $\mathbf{q}(\mathbf{r}, \tau)$ is a superposition of plane waves, but depending on the symmetry of the problem Bessel functions or spherical wave functions are to be chosen as the eigenfunctions.

The vector \mathbf{F} contains nonlinear functions P_{i1} . They need to be expanded in a Taylor series in powers of the components of vector \mathbf{q} . We restrict our consideration to quadratic terms in the expansion of P_{i1} . It is easily shown, that the terms of this order are necessary to obtain the dispersion equation for the averaged amplitudes of the unstable modes with an accuracy to terms quadratic in the fluctuation intensity. As a result, the components of vector \mathbf{F} containing random fields take the form:

$$F_i = f_{i1}(\mathbf{r}, \tau) \left(p_i^{(0)} + \sum_{\mu=1}^2 p_{i,\mu}^{(1)} q_\mu + \sum_{\mu,\nu=1}^2 p_{i,\mu\nu}^{(2)} q_\mu q_\nu \right),$$

where $p_i^{(0)} = P_{i1}(x_{10}, x_{20})$, $p_{i,\mu}^{(1)} = \left. \frac{\partial P_{i1}}{\partial x_\mu} \right|_{x_{10}, x_{20}}$, $p_{i,\mu\nu}^{(2)} = \frac{1}{2!} \left. \frac{\partial^2 P_{i1}}{\partial x_\mu \partial x_\nu} \right|_{x_{10}, x_{20}}$.

Unstable modes lie in a narrow band of wavenumbers defining the instability region of the system. This makes it possible to construct wave packets by summing over the wave vectors, which lie in a small interval. Thus, carrying modes with discrete values of wave vectors and slowly varying amplitudes $\xi_{\mathbf{k}}^{(j)}(\tau)$ are chosen [29].

To obtain the equations for the mode amplitudes $\xi_{\mathbf{k}}^{(j)}(\tau)$ we substitute eq. (9) in eq. (5), multiply the equation derived on the left by $\exp(-i\mathbf{k}\mathbf{r}) \mathbf{O}^{*(j')}(\mathbf{k})$ and integrate over the region, which is much greater than the oscillation period $\exp(i\mathbf{k}\mathbf{r})$, but in which $\xi_{\mathbf{k}}^{(j)}(\tau)$ varies very little. Here $\mathbf{O}^{*(j')}(\mathbf{k})$ are eigenvectors of the operator conjugate to $K(\nabla^2)$: $\mathbf{O}^{(j)} \mathbf{O}^{*(j')} = \delta_{jj'}$.

After transformations, the system of equations for the amplitudes of the modes takes the form

$$\begin{aligned}
\frac{d\xi_{\mathbf{k}}^{(j)}}{d\tau} - \lambda_j(\mathbf{k})\xi_{\mathbf{k}}^{(j)} &= \sum_{j'j'',\mathbf{k}'\mathbf{k}''} \sigma_{j'j''}^{(j)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'') \xi_{\mathbf{k}'}^{(j')} \xi_{\mathbf{k}''}^{(j'')} \delta(\mathbf{k}' + \mathbf{k}'' - \mathbf{k}) + \\
&+ \sum_{j'j''j''',\mathbf{k}'\mathbf{k}''\mathbf{k}'''} \sigma_{j'j''j'''}^{(j)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'', \mathbf{k}''') \xi_{\mathbf{k}'}^{(j')} \xi_{\mathbf{k}''}^{(j'')} \xi_{\mathbf{k}'''}^{(j''')} \delta(\mathbf{k}' + \mathbf{k}'' + \mathbf{k}''' - \mathbf{k}) + \sum_{\varphi=1}^2 O_{\varphi}^{*(j)}(\mathbf{k}) p_{\varphi}^{(0)} z_{\varphi,\mathbf{k}}(\tau) + \\
&+ \sum_{\varphi=1}^2 \sum_{j',\mathbf{k}'} \varepsilon_{\varphi,j'}^{(j)}(\mathbf{k}, \mathbf{k}') \xi_{\mathbf{k}'}^{(j')} z_{\varphi,\mathbf{k}-\mathbf{k}'}(\tau) + \sum_{\varphi=1}^2 \sum_{j'j'',\mathbf{k}'\mathbf{k}''} \varepsilon_{\varphi,j'j''}^{(j)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'') \xi_{\mathbf{k}'}^{(j')} \xi_{\mathbf{k}''}^{(j'')} z_{\varphi,\mathbf{k}-\mathbf{k}'-\mathbf{k}''}(\tau).
\end{aligned} \tag{10}$$

Functions $\sigma_{j'j''}^{(j)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'')$, $\sigma_{j'j''j'''}^{(j)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'', \mathbf{k}''')$, $\varepsilon_{\varphi,j'}^{(j)}(\mathbf{k}, \mathbf{k}')$, $\varepsilon_{\varphi,j'j''}^{(j)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'')$ introduced in equations (10) are presented in the Appendix A.

Random processes $z_{\varphi,\mathbf{k}}(\tau) = \int f_{\varphi 1}(\mathbf{r}, \tau) e^{-i\mathbf{k}\mathbf{r}} d\mathbf{r}$ are components of the random vector field $\mathbf{z}(\tau)$ with zero mean, φ and \mathbf{k} are index arguments of this field.

Assuming, that the correlation time of random fields is considerably smaller than all the characteristic times of the deterministic problem (4), correlation functions for the components of field $\mathbf{z}(\tau)$ will have the form: $K[z_{j,\mathbf{k}}(t), z_{l,\mathbf{k}'}(\tau)] = g_{jl}(|\mathbf{k}|) \delta(\mathbf{k} - \mathbf{k}') \delta(t - \tau) \delta_{jl}$. Further we assume for definiteness

$$\Phi_j(|\mathbf{r} - \mathbf{r}'|) = \theta_j \exp(-k_{fj}|\mathbf{r} - \mathbf{r}'|). \tag{11}$$

Here θ_j are noise intensity, k_{fj} are magnitudes inverse to correlation length. For two-dimensional media $g_{jj} = \theta_j k_{fj} / [2\pi^2(k^2 + k_{fj}^2)^{-3/2}]$.

The system (10) contains both stable and unstable modes. In the vicinity of the bifurcation point the relaxation times of unstable modes are considerably greater than the relaxation times of stable modes, therefore the latter adiabatically follow the former ones. This makes it possible to exclude the stable modes from the equations (10). To perform the procedure of adiabatic elimination [29, 30] of stable modes, we rewrite the system of equations (10) dividing it into two subsystems of equations for the unstable modes (denote them by an additional index (u)) and for the stable modes (s).

Since the unstable modes can grow to infinity if we neglect the nonlinear terms, we write equations for them with an accuracy of cubic terms, which provide nonlinear stabilization of the instability (if the cubic terms are not enough, it is necessary to take into account the fifth orders).

We assume that the amplitudes of stable modes are significantly smaller than the amplitudes of unstable modes $|\xi_s| \ll |\xi_u|$ and their variations occur self-consistently: $\xi_s \sim \xi_u^2$. In addition, $z_{\varphi,\mathbf{k}} \sim \xi_s$.

In the equations for the stable mode amplitudes we keep only the terms necessary to obtain the equations for the unstable mode amplitudes with an accuracy to third-order terms. Then from eq. (10) for the stable mode amplitudes we obtain the following equation:

$$\begin{aligned}
\frac{d\xi_{\mathbf{k}s}^{(j)}}{d\tau} - \lambda_j(\mathbf{k}_s)\xi_{\mathbf{k}s}^{(j)} &= \sum_{\mathbf{k}'u\mathbf{k}''u} \sigma_{11}^{(j)}(\mathbf{k}_s, \mathbf{k}'_u, \mathbf{k}''_u) \xi_{\mathbf{k}'u}^{(1)} \xi_{\mathbf{k}''u}^{(1)} \delta(\mathbf{k}'_u + \mathbf{k}''_u - \mathbf{k}_s) + \\
&+ \sum_{\varphi=1}^2 O_{\varphi}^{*(j)}(\mathbf{k}_s) p_{\varphi}^{(0)} z_{\varphi,\mathbf{k}s}(\tau) + \sum_{\varphi=1}^2 \sum_{\mathbf{k}'u} \varepsilon_{\varphi,1}^{(j)}(\mathbf{k}_s, \mathbf{k}'_u) \xi_{\mathbf{k}'u}^{(1)} z_{\varphi,\mathbf{k}s-\mathbf{k}'u}(\tau) + \\
&+ \sum_{\varphi=1}^2 \sum_{\mathbf{k}'u\mathbf{k}''u} \varepsilon_{\varphi,11}^{(j)}(\mathbf{k}_s, \mathbf{k}'_u, \mathbf{k}''_u) \xi_{\mathbf{k}'u}^{(1)} \xi_{\mathbf{k}''u}^{(1)} z_{\varphi,\mathbf{k}s-\mathbf{k}'u-\mathbf{k}''u}(\tau).
\end{aligned} \tag{12}$$

The equations for the unstable mode amplitudes have the form:

$$\begin{aligned}
\frac{d\xi_{\mathbf{k}u}^{(1)}}{d\tau} - \lambda_1(\mathbf{k}_u)\xi_{\mathbf{k}u}^{(1)} &= \sum_{\mathbf{k}'u\mathbf{k}''u} \sigma_{11}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}''_u) \xi_{\mathbf{k}'u}^{(1)} \xi_{\mathbf{k}''u}^{(1)} \delta(\mathbf{k}'_u + \mathbf{k}''_u - \mathbf{k}_u) + \\
&+ \sum_{\mathbf{k}'u\mathbf{k}''u\mathbf{k}'''u} \sigma_{111}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}''_u, \mathbf{k}'''_u) \xi_{\mathbf{k}'u}^{(1)} \xi_{\mathbf{k}''u}^{(1)} \xi_{\mathbf{k}'''u}^{(1)} \delta(\mathbf{k}'_u + \mathbf{k}''_u + \mathbf{k}'''_u - \mathbf{k}_u) + \\
&+ \sum_{\varphi=1}^2 O_{\varphi}^{*(1)}(\mathbf{k}_u) p_{\varphi}^{(0)} z_{\varphi, \mathbf{k}u}(\tau) + \sum_{\varphi=1}^2 \sum_{\mathbf{k}'u} \varepsilon_{\varphi, 1}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u) \xi_{\mathbf{k}'u}^{(1)} z_{\varphi, \mathbf{k}u - \mathbf{k}'u}(\tau) + \\
&+ \sum_{\varphi=1}^2 \sum_{\mathbf{k}'u\mathbf{k}''u} \varepsilon_{\varphi, 11}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}''_u) \xi_{\mathbf{k}'u}^{(1)} \xi_{\mathbf{k}''u}^{(1)} z_{\varphi, \mathbf{k}u - \mathbf{k}'u - \mathbf{k}''u}(\tau) + \\
&+ \sum_{\psi, \varphi=1}^2 \sum_{\mathbf{k}s} \varepsilon_{\varphi, \psi}^{(1)}(\mathbf{k}_u, \mathbf{k}_s) \xi_{\mathbf{k}s}^{(\psi)} z_{\varphi, \mathbf{k}u - \mathbf{k}s}(\tau) + \\
&+ \sum_{\psi=1}^2 \sum_{\mathbf{k}'u\mathbf{k}s} \left[\sigma_{1\psi}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}_s) + \sigma_{\psi 1}^{(1)}(\mathbf{k}_u, \mathbf{k}_s, \mathbf{k}'_u) \right] \xi_{\mathbf{k}'u}^{(1)} \xi_{\mathbf{k}s}^{(\psi)} \delta(\mathbf{k}'_u + \mathbf{k}_s - \mathbf{k}_u) + \\
&+ \sum_{\psi, \varphi=1}^2 \sum_{\mathbf{k}'u\mathbf{k}s} \left[\varepsilon_{\varphi, 1\psi}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}_s) + \varepsilon_{\varphi, \psi 1}^{(1)}(\mathbf{k}_u, \mathbf{k}_s, \mathbf{k}'_u) \right] \xi_{\mathbf{k}'u}^{(1)} \xi_{\mathbf{k}s}^{(\psi)} z_{\varphi, \mathbf{k}u - \mathbf{k}'u - \mathbf{k}s}(\tau).
\end{aligned} \tag{13}$$

Neglecting the time derivative $d\xi_{\mathbf{k}s}^{(j)}/d\tau$ in equations (12) [29], expressing the amplitudes $\xi_{\mathbf{k}s}^{(j)}$ from them and substituting the latter in eq. (13), we obtain a system of equations for the unstable mode amplitudes $\xi_{\mathbf{k}u}^{(1)}$:

$$\begin{aligned}
\frac{d\xi_{\mathbf{k}u}^{(1)}}{d\tau} &= F_{\mathbf{k}u}(\tau), \\
F_{\mathbf{k}u}(\tau) &= \lambda_1(\mathbf{k}_u)\xi_{\mathbf{k}u}^{(1)} + \sum_{\mathbf{k}'u\mathbf{k}''u} \sigma_{11}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}''_u) \xi_{\mathbf{k}'u}^{(1)} \xi_{\mathbf{k}''u}^{(1)} \delta(\mathbf{k}'_u + \mathbf{k}''_u - \mathbf{k}_u) + \\
&+ \sum_{\mathbf{k}'u\mathbf{k}''u\mathbf{k}'''u} \omega(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}''_u, \mathbf{k}'''_u, \mathbf{k}_u - \mathbf{k}'_u) \xi_{\mathbf{k}'u}^{(1)} \xi_{\mathbf{k}''u}^{(1)} \xi_{\mathbf{k}'''u}^{(1)} \delta(\mathbf{k}'_u + \mathbf{k}''_u + \mathbf{k}'''_u - \mathbf{k}_u) + \\
&+ \sum_{\varphi=1}^2 O_{\varphi}^{*(1)}(\mathbf{k}_u) p_{\varphi}^{(0)} z_{\varphi, \mathbf{k}u}(\tau) - \sum_{\psi, \varphi, \varphi'=1}^2 \sum_{\mathbf{k}s} \zeta_{\varphi, \psi, \varphi'}(\mathbf{k}_u, \mathbf{k}_s) z_{\varphi, \mathbf{k}u - \mathbf{k}s}(\tau) z_{\varphi', \mathbf{k}s}(\tau) + \\
&+ \sum_{\varphi=1}^2 \sum_{\mathbf{k}'u} \eta_{\varphi}(\mathbf{k}_u, \mathbf{k}'_u) \xi_{\mathbf{k}'u}^{(1)} z_{\varphi, \mathbf{k}u - \mathbf{k}'u}(\tau) + \sum_{\varphi=1}^2 \sum_{\mathbf{k}'u\mathbf{k}''u} \nu_{\varphi}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}''_u) \xi_{\mathbf{k}'u}^{(1)} \xi_{\mathbf{k}''u}^{(1)} z_{\varphi, \mathbf{k}u - \mathbf{k}'u - \mathbf{k}''u}(\tau) - \\
&- \sum_{\psi, \varphi, \varphi'=1}^2 \sum_{\mathbf{k}s, \mathbf{k}'u} A_{\varphi, \psi, \varphi'}(\mathbf{k}_u, \mathbf{k}_s, \mathbf{k}'_u) \xi_{\mathbf{k}'u}^{(1)} z_{\varphi', \mathbf{k}s - \mathbf{k}'u}(\tau) z_{\varphi, \mathbf{k}u - \mathbf{k}s}(\tau) - \\
&- \sum_{\psi, \varphi, \varphi'=1}^2 \sum_{\mathbf{k}'u\mathbf{k}s} B_{\varphi, \psi, \varphi'}(\mathbf{k}_u, \mathbf{k}_s, \mathbf{k}'_u) \xi_{\mathbf{k}'u}^{(1)} z_{\varphi, \mathbf{k}u - \mathbf{k}'u - \mathbf{k}s}(\tau) z_{\varphi', \mathbf{k}s}(\tau) - \\
&- \sum_{\psi, \varphi, \varphi'=1}^2 \sum_{\mathbf{k}'u\mathbf{k}''u\mathbf{k}s} C_{\varphi, \psi, \varphi'}(\mathbf{k}_u, \mathbf{k}_s, \mathbf{k}'_u, \mathbf{k}''_u) \xi_{\mathbf{k}'u}^{(1)} \xi_{\mathbf{k}''u}^{(1)} z_{\varphi', \mathbf{k}s - \mathbf{k}'u - \mathbf{k}''u}(\tau) z_{\varphi, \mathbf{k}u - \mathbf{k}s}(\tau) -
\end{aligned} \tag{14}$$

$$\begin{aligned}
& - \sum_{\psi, \varphi'=1}^2 \sum_{\mathbf{k}'_u \mathbf{k}''_u \mathbf{k}_s} D_{\psi, \psi, \varphi'}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}_s, \mathbf{k}''_u) \xi_{\mathbf{k}'_u}^{(1)} \xi_{\mathbf{k}''_u}^{(1)} z_{\varphi, \mathbf{k}_u - \mathbf{k}'_u - \mathbf{k}_s}(\tau) z_{\varphi', \mathbf{k}_s - \mathbf{k}''_u}(\tau) - \\
& - \sum_{\psi, \varphi=1}^2 \sum_{\mathbf{k}'_u \mathbf{k}''_u \mathbf{k}'''_u} E_{\psi, \varphi}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}_u - \mathbf{k}'_u, \mathbf{k}''_u, \mathbf{k}'''_u) \xi_{\mathbf{k}'_u}^{(1)} \xi_{\mathbf{k}''_u}^{(1)} \xi_{\mathbf{k}'''_u}^{(1)} z_{\varphi, \mathbf{k}_u - \mathbf{k}'_u - \mathbf{k}''_u - \mathbf{k}'''_u}(\tau) - \\
& - \sum_{\psi, \varphi=1}^2 \sum_{\mathbf{k}'_u \mathbf{k}''_u \mathbf{k}'''_u} F_{\psi, \varphi}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}''_u, \mathbf{k}'''_u) \xi_{\mathbf{k}'_u}^{(1)} \xi_{\mathbf{k}''_u}^{(1)} \xi_{\mathbf{k}'''_u}^{(1)} z_{\varphi, \mathbf{k}_u - \mathbf{k}'_u - \mathbf{k}''_u - \mathbf{k}'''_u}(\tau) - \\
& - \sum_{\psi, \varphi'=1}^2 \sum_{\mathbf{k}'_u \mathbf{k}''_u \mathbf{k}'''_u \mathbf{k}_s} G_{\psi, \varphi, \varphi'}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}_s, \mathbf{k}''_u, \mathbf{k}'''_u) \xi_{\mathbf{k}'_u}^{(1)} \xi_{\mathbf{k}''_u}^{(1)} \xi_{\mathbf{k}'''_u}^{(1)} z_{\varphi, \mathbf{k}_u - \mathbf{k}'_u - \mathbf{k}_s}(\tau) z_{\varphi', \mathbf{k}_s - \mathbf{k}''_u - \mathbf{k}'''_u}(\tau).
\end{aligned}$$

Function $\omega(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}''_u, \mathbf{k}'''_u, \mathbf{k}_u - \mathbf{k}'_u)$ and others introduced in equations (14) are presented in the Appendix B. The modes $\xi_{\mathbf{k}_u}^{(1)}$ are order parameters. Their collaboration or competition determines the behavior of the system.

The system of equations (14) is still difficult to analyze because it contains random components. Further analysis of the equations (14) may consist in their statistical averaging or in obtaining the Fokker-Planck equation.

III. STATISTICAL AVERAGING

For statistical averaging we used the relationship between the moments and the correlation functions [37] and multi-dimensional generalization of Furutsu - Novikov formula [31]. Taking into account the formal solution of Eqs. (14) and assuming, that we can neglect higher than second order correlation functions, it can be shown, that in Eqs. (14) the terms containing the product $\xi_{\mathbf{k}'_u}^{(1)} \xi_{\mathbf{k}''_u}^{(1)} \xi_{\mathbf{k}'''_u}^{(1)} z_{\varphi, \mathbf{k}_1}(\tau) z_{\varphi', \mathbf{k}_2}(\tau)$ and $\xi_{\mathbf{k}'_u}^{(1)} \xi_{\mathbf{k}''_u}^{(1)} \xi_{\mathbf{k}'''_u}^{(1)} z_{\varphi, \mathbf{k}}(\tau)$ must be discarded since they do not contribute to the average values of the modes $\xi_{\mathbf{k}_u}^{(1)}$ in averaging. We note here, that the procedure of correlation splitting leads to the appearance of a similar correlation for the other interacting modes. Therefore, this procedure should be performed until all the terms containing the necessary degree of intensity fluctuations are taken into account. The remaining correlation functions can be neglected owing to their smallness, as the terms obtained after their splitting are proportional to a higher degree of noise intensity.

In order to obtain after averaging corrections to the increments of unstable mode amplitudes with an accuracy to terms quadratic in the noise intensity, when calculating the functional derivatives in the Furutsu - Novikov formula it is necessary to retain the terms containing the product $\xi_{\mathbf{k}'_u}^{(1)} z_{\varphi, \mathbf{k}}(\tau)$.

Below is given only the structure of equations obtained by the procedure of averaging since they have a very complicated form:

$$\begin{aligned}
& \frac{d\langle \xi_{\mathbf{k}_u}^{(1)} \rangle}{d\tau} - \lambda_1(\mathbf{k}_u) \langle \xi_{\mathbf{k}_u}^{(1)} \rangle = \mathbf{L}_0(\mathbf{k}_u) + \mathbf{L}_1(\mathbf{k}_u, \mathbf{k}'_u) \langle \xi_{\mathbf{k}'_u}^{(1)} \rangle + \\
& + \mathbf{L}_2(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}''_u, \mathbf{k}'''_u, \mathbf{k}_s) \langle \xi_{\mathbf{k}'_u}^{(1)} \xi_{\mathbf{k}''_u}^{(1)} \rangle + \mathbf{L}_3(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}''_u, \mathbf{k}'''_u, \mathbf{k}_u^{IV}, \mathbf{k}_s, \mathbf{k}'_s) \langle \xi_{\mathbf{k}'_u}^{(1)} \xi_{\mathbf{k}''_u}^{(1)} \xi_{\mathbf{k}^{IV}_u}^{(1)} \rangle.
\end{aligned} \tag{15}$$

The analysis of the equations (15) leads to the following conclusions.

First, after the averaging of Eqs. (14) additional terms, that do not depend on $\langle \xi_{\mathbf{k}_u}^{(1)} \rangle$, arise in (15). They are determined by the parameters of the problem, the type of the correlation function g_{jj} , the noise intensity, and the wavenumber of a given mode.

Second, in the system (15) there are additional terms proportional to $\langle \xi_{\mathbf{k}u}^{(1)} \rangle$. This leads to a variation of eigenvalues of unstable mode amplitudes in comparison with the deterministic case. As a result, the region of instability of the system, the beginning of the process of destruction of a statistically stationary homogeneous state and pattern formation as well as the duration of the transitional regime from one statistically stationary state to another are changed.

Selecting from $\mathbf{L}_1(\mathbf{k}_u, \mathbf{k}'_u)$ the terms that contribute to the increment of $\langle \xi_{\mathbf{k}u}^{(1)} \rangle$, we obtain the dispersion equation

$$\lambda = \lambda_1(\mathbf{k}_u) + \mathbf{L}_1(\mathbf{k}_u, \mathbf{k}_u) + \mathbf{L}_1(\mathbf{k}_u, -\mathbf{k}_u). \quad (16)$$

When deriving eq.(16) we take into account the, fact that $\xi_{-\mathbf{k}u}^{(1)} = \xi_{\mathbf{k}u}^{*(1)} = \xi_{\mathbf{k}u}^{(1)}$, because the solutions of equations (14) must be real. From equation (16) and expressions $\mathbf{L}_1(\mathbf{k}_u, \mathbf{k}'_u)$ and g_{ii} , it follows, that increments ($\text{Re } \lambda$) of the unstable mode averaged amplitudes are proportional to the intensity of noise and depend on the correlation length. Herewith, the intensity of noise becomes another bifurcation parameter and patterns begin to form, when the parameter $\chi_{2\theta}$ is different from χ_{2c} . Research of particular systems shows, that the value of the bifurcation parameter χ_2 is shifted to the subcritical region for the case of deterministic description. Finally, we note, that in the external random fields pattern formation occurs due to multimode interactions, whereby the conditions the resonant interaction of the modes are also different from the deterministic case. In particular, interaction between different configurations of modes with the modes with doubled wavenumbers arises, for example $2\mathbf{k}'_u - \mathbf{k}''_u = \mathbf{k}_u$, as well as the five-mode interaction.

Thus, the above theoretical analysis allows us to describe the evolution of the stochastic systems under consideration, in the vicinity of the Turing bifurcation point in more detail.

IV. FOKKER-PLANCK EQUATION FOR THE ORDER PARAMETERS

If the system parameters are such that it is in the supercritical region then with increasing noise intensity, the system will "go" farther and farther from the bifurcation point of a deterministic system. To describe the state of the system in this case we can use the Fokker-Planck equation.

For the system (14) the Fokker - Planck equation can be written in general form as follows [35]:

$$\begin{aligned} \frac{\partial w(\{\xi_{\mathbf{k}u}^{(1)}\}, \tau)}{\partial \tau} = & - \sum_{\mathbf{k}u} \frac{\partial}{\partial \xi_{\mathbf{k}u}^{(1)}} \left\{ \left(\langle F_{\mathbf{k}u}(\tau) \rangle + \sum_{\mathbf{q}u} \int_{-\infty}^0 K \left[\frac{\partial F_{\mathbf{k}u}(\tau)}{\partial \xi_{\mathbf{q}u}^{(1)}}, F_{\mathbf{q}u}(t') \right] dt' \right) w \right\} + \\ & + \sum_{\mathbf{k}u, \mathbf{q}u} \frac{\partial^2}{\partial \xi_{\mathbf{k}u}^{(1)} \partial \xi_{\mathbf{q}u}^{(1)}} \left\{ \left(\int_{-\infty}^0 K[F_{\mathbf{k}u}(\tau), F_{\mathbf{q}u}(t')] dt' \right) w \right\}. \end{aligned} \quad (17)$$

Here $w(\{\xi_{\mathbf{k}u}^{(1)}\}, \tau)$ is multivariate probability density, which determines the probability of some configuration of unstable modes $\{\xi_{\mathbf{k}u}^{(1)}\}$. After transformation with an accuracy to terms linear in the noise intensity we can obtain the correlation functions appearing in (17):

$$\begin{aligned}
K\left[\frac{\partial F_{\mathbf{k}u}(\tau)}{\partial \xi_{\mathbf{q}u}^{(1)}}, F_{\mathbf{q}u}(t')\right] &= \sum_{\varphi} \eta_{\varphi}(\mathbf{k}_u, \mathbf{q}_u) O_{\varphi}^{*(1)}(\mathbf{q}_u) p_{\varphi}^{(0)} g_{\varphi\varphi}(|\mathbf{k}_u - \mathbf{q}_u|) \delta_{\mathbf{k}u - \mathbf{q}u, \mathbf{q}u} \delta(\tau - t') + \\
&+ \sum_{\varphi} [\nu_{\varphi}(\mathbf{k}_u, \mathbf{q}_u, \mathbf{k}_u - 2\mathbf{q}_u) + \nu_{\varphi}(\mathbf{k}_u, \mathbf{k}_u - 2\mathbf{q}_u, \mathbf{q}_u)] O_{\varphi}^{*(1)}(\mathbf{q}_u) p_{\varphi}^{(0)} g_{\varphi\varphi}(|\mathbf{q}_u|) \xi_{\mathbf{k}u - 2\mathbf{q}u} \delta(\tau - t') + \\
&+ \sum_{\varphi} \eta_{\varphi}(\mathbf{k}_u, \mathbf{q}_u) \eta_{\varphi}(\mathbf{q}_u, 2\mathbf{q}_u - \mathbf{k}_u) g_{\varphi\varphi}(|\mathbf{k}_u - \mathbf{q}_u|) \xi_{2\mathbf{q}u - \mathbf{k}u} \delta(\tau - t') + \\
&+ \sum_{\varphi, \mathbf{q}'u} \eta_{\varphi}(\mathbf{k}_u, \mathbf{q}_u) \nu_{\varphi}(\mathbf{q}_u, \mathbf{q}'u, 2\mathbf{q}_u - \mathbf{k}_u - \mathbf{q}'u) g_{\varphi\varphi}(|\mathbf{k}_u - \mathbf{q}_u|) \xi_{\mathbf{q}'u} \xi_{2\mathbf{q}u - \mathbf{k}u - \mathbf{q}'u} \delta(\tau - t') + \\
&+ \sum_{\varphi, \mathbf{q}'u, \mathbf{q}''u} [\nu_{\varphi}(\mathbf{k}_u, \mathbf{q}_u, \mathbf{k}_u - 2\mathbf{q}_u + \mathbf{q}'u + \mathbf{q}''u) + \nu_{\varphi}(\mathbf{k}_u, \mathbf{k}_u - 2\mathbf{q}_u + \mathbf{q}'u + \mathbf{q}''u, \mathbf{q}_u)] \times \\
&\quad \times \nu_{\varphi}(\mathbf{q}_u, \mathbf{q}'u, \mathbf{q}''u) g_{\varphi\varphi}(|\mathbf{q}_u - \mathbf{q}'u - \mathbf{q}''u|) \xi_{\mathbf{q}'u} \xi_{\mathbf{q}''u} \xi_{\mathbf{k}u - 2\mathbf{q}u + \mathbf{q}'u + \mathbf{q}''u} \delta(\tau - t') + \\
&+ \sum_{\varphi, \mathbf{q}'u} \eta_{\varphi}(\mathbf{q}_u, \mathbf{q}'u) [\nu_{\varphi}(\mathbf{k}_u, \mathbf{k}_u - 2\mathbf{q}_u + \mathbf{q}'u, \mathbf{q}_u) + \nu_{\varphi}(\mathbf{k}_u, \mathbf{q}_u, \mathbf{k}_u - 2\mathbf{q}_u + \mathbf{q}'u)] \times \\
&\quad \times g_{\varphi\varphi}(|\mathbf{q}_u - \mathbf{q}'u|) \xi_{\mathbf{q}'u} \xi_{\mathbf{k}u - 2\mathbf{q}u + \mathbf{q}'u} \delta(\tau - t'), \\
K[F_{\mathbf{k}u}(\tau), F_{\mathbf{q}u}(t')] &= \sum_{\varphi} [O_{\varphi}^{*(1)}(\mathbf{k}_u)]^2 (p_{\varphi}^{(0)})^2 g_{\varphi\varphi}(|\mathbf{k}_u|) \delta_{\mathbf{k}u, \mathbf{q}u} \delta(\tau - t') + \\
&+ \sum_{\varphi} \eta_{\varphi}(\mathbf{k}_u, \mathbf{k}_u - \mathbf{q}_u) O_{\varphi}^{*(1)}(\mathbf{q}_u) p_{\varphi}^{(0)} g_{\varphi\varphi}(|\mathbf{q}_u|) \xi_{\mathbf{k}u - \mathbf{q}u} \delta(\tau - t') + \\
&+ \sum_{\varphi, \mathbf{k}'u} \nu_{\varphi}(\mathbf{k}_u, \mathbf{k}'u, \mathbf{k}_u - \mathbf{k}'u - \mathbf{q}_u) O_{\varphi}^{*(1)}(\mathbf{q}_u) p_{\varphi}^{(0)} g_{\varphi\varphi}(|\mathbf{q}_u|) \xi_{\mathbf{k}'u} \xi_{\mathbf{k}u - \mathbf{k}'u - \mathbf{q}u} \delta(\tau - t') + \\
&+ \sum_{\varphi} \eta_{\varphi}(\mathbf{q}_u, \mathbf{q}_u - \mathbf{k}_u) O_{\varphi}^{*(1)}(\mathbf{k}_u) p_{\varphi}^{(0)} g_{\varphi\varphi}(|\mathbf{k}_u|) \xi_{\mathbf{q}u - \mathbf{k}u} \delta(\tau - t') + \\
&+ \sum_{\varphi, \mathbf{q}'u} \eta_{\varphi}(\mathbf{k}_u, \mathbf{k}_u - \mathbf{q}_u + \mathbf{q}'u) \eta_{\varphi}(\mathbf{q}_u, \mathbf{q}'u) g_{\varphi\varphi}(|\mathbf{q}_u - \mathbf{q}'u|) \xi_{\mathbf{q}'u} \xi_{\mathbf{k}u - \mathbf{q}u + \mathbf{q}'u} \delta(\tau - t') + \\
&+ \sum_{\varphi, \mathbf{k}'u, \mathbf{q}'u} \nu_{\varphi}(\mathbf{k}_u, \mathbf{k}'u, \mathbf{k}_u - \mathbf{k}'u - \mathbf{q}_u + \mathbf{q}'u) \eta_{\varphi}(\mathbf{q}_u, \mathbf{q}'u) g_{\varphi\varphi}(|\mathbf{q}_u - \mathbf{q}'u|) \xi_{\mathbf{k}'u} \xi_{\mathbf{q}'u} \xi_{\mathbf{k}u - \mathbf{k}'u - \mathbf{q}u + \mathbf{q}'u} \delta(\tau - t') + \\
&+ \sum_{\varphi, \mathbf{q}'u} \nu_{\varphi}(\mathbf{q}_u, \mathbf{q}'u, \mathbf{q}_u - \mathbf{q}'u - \mathbf{k}_u) O_{\varphi}^{*(1)}(\mathbf{k}_u) p_{\varphi}^{(0)} g_{\varphi\varphi}(|\mathbf{k}_u|) \xi_{\mathbf{q}'u} \xi_{\mathbf{q}u - \mathbf{q}'u - \mathbf{k}u} \delta(\tau - t') + \\
&+ \sum_{\varphi, \mathbf{k}'u, \mathbf{q}'u} \nu_{\varphi}(\mathbf{q}_u, \mathbf{q}'u, \mathbf{q}_u - \mathbf{q}'u - \mathbf{k}_u + \mathbf{k}'u) \eta_{\varphi}(\mathbf{k}_u, \mathbf{k}'u) g_{\varphi\varphi}(|\mathbf{k}_u - \mathbf{k}'u|) \xi_{\mathbf{k}'u} \xi_{\mathbf{q}'u} \xi_{\mathbf{q}u - \mathbf{q}'u - \mathbf{k}u + \mathbf{k}'u} \delta(\tau - t') + \\
&+ \sum_{\varphi, \mathbf{k}'u, \mathbf{q}'u, \mathbf{q}''u} \nu_{\varphi}(\mathbf{k}_u, \mathbf{k}'u, \mathbf{k}_u - \mathbf{k}'u - \mathbf{q}_u + \mathbf{q}'u + \mathbf{q}''u) \nu_{\varphi}(\mathbf{q}_u, \mathbf{q}'u, \mathbf{q}''u) \times \\
&\quad \times g_{\varphi\varphi}(|\mathbf{q}_u - \mathbf{q}'u - \mathbf{q}''u|) \xi_{\mathbf{k}'u} \xi_{\mathbf{q}'u} \xi_{\mathbf{q}''u} \xi_{\mathbf{k}u - \mathbf{k}'u - \mathbf{q}u + \mathbf{q}'u + \mathbf{q}''u} \delta(\tau - t').
\end{aligned}$$

Suppose, that the space of the system under study is two-dimensional. If in this space there is only one unstable mode with the wave vector \mathbf{k}_c and amplitude $\xi_{\mathbf{k}c}$ the equation (17) is significantly simplified

$$\frac{\partial w(\xi_{\mathbf{k}c}, \tau)}{\partial \tau} = -\frac{\partial}{\partial \xi_{\mathbf{k}c}} \left[(a\xi_{\mathbf{k}c} + b\xi_{\mathbf{k}c}^3)w - (c + d\xi_{\mathbf{k}c}^2 + e\xi_{\mathbf{k}c}^4) \frac{\partial w}{\partial \xi_{\mathbf{k}c}} \right]. \quad (18)$$

Here

$$a = \lambda_1(\mathbf{k}_c) - \frac{1}{2} \sum_{\varphi} \eta_{\varphi}^2(\mathbf{k}_c, \mathbf{k}_c) g_{\varphi\varphi}(0) - \sum_{\varphi} \nu_{\varphi}(\mathbf{k}_c, \mathbf{k}_c, \mathbf{k}_c) O_{\varphi}^{*(1)}(\mathbf{k}_c) p_{\varphi}^{(0)} g_{\varphi\varphi}(\mathbf{k}_c),$$

$$b = \omega(\mathbf{k}_c, \mathbf{k}_c, \mathbf{k}_c, \mathbf{k}_c, 0) - \sum_{\varphi} \nu_{\varphi}^2(\mathbf{k}_c, \mathbf{k}_c, \mathbf{k}_c) g_{\varphi\varphi}(\mathbf{k}_c),$$

$$c = \frac{1}{2} \sum_{\varphi} \left[O_{\varphi}^{*(1)}(\mathbf{k}_c) p_{\varphi}^{(0)} \right]^2 g_{\varphi\varphi}(\mathbf{k}_c),$$

$$d = \frac{1}{2} \sum_{\varphi} \eta_{\varphi}^2(\mathbf{k}_c, \mathbf{k}_c) g_{\varphi\varphi}(0) + \sum_{\varphi} \nu_{\varphi}(\mathbf{k}_c, \mathbf{k}_c, \mathbf{k}_c) O_{\varphi}^{*(1)}(\mathbf{k}_c) p_{\varphi}^{(0)} g_{\varphi\varphi}(\mathbf{k}_c),$$

$$e = \frac{1}{2} \sum_{\varphi} \nu_{\varphi}^2(\mathbf{k}_c, \mathbf{k}_c, \mathbf{k}_c) g_{\varphi\varphi}(\mathbf{k}_c).$$

The stationary solution of equation (18) has the form:

$$\begin{aligned} w(\xi_{\mathbf{k}c}) &= N |c + d\xi_{\mathbf{k}c}^2 + e\xi_{\mathbf{k}c}^4|^{\frac{b}{4e}} \left| \frac{2e\xi_{\mathbf{k}c}^2 + d - \sqrt{d^2 - 4ec}}{2e\xi_{\mathbf{k}c}^2 + d + \sqrt{d^2 - 4ec}} \right|^{\frac{2ae - bd}{4e\sqrt{d^2 - 4ec}}}, \quad d^2 > 4ec, \\ &= N |c + d\xi_{\mathbf{k}c}^2 + e\xi_{\mathbf{k}c}^4|^{\frac{b}{4e}} \exp \left[\frac{2ae - bd}{2e\sqrt{4ec - d^2}} \arctg \left(\frac{2e\xi_{\mathbf{k}c}^2 + d}{\sqrt{4ec - d^2}} \right) \right], \quad 4ec > d^2. \end{aligned} \quad (19)$$

Here N is the normalization constant:

$$\begin{aligned} N &= 1 / \int_{-\infty}^{+\infty} |c + d\xi_{\mathbf{k}c}^2 + e\xi_{\mathbf{k}c}^4|^{\frac{b}{4e}} \left| \frac{2e\xi_{\mathbf{k}c}^2 + d - \sqrt{d^2 - 4ec}}{2e\xi_{\mathbf{k}c}^2 + d + \sqrt{d^2 - 4ec}} \right|^{\frac{2ae - bd}{4e\sqrt{d^2 - 4ec}}} d\xi_{\mathbf{k}c}, \quad d^2 > 4ec, \\ &= 1 / \int_{-\infty}^{+\infty} |c + d\xi_{\mathbf{k}c}^2 + e\xi_{\mathbf{k}c}^4|^{\frac{b}{4e}} \exp \left[\frac{2ae - bd}{2e\sqrt{4ec - d^2}} \arctg \left(\frac{2e\xi_{\mathbf{k}c}^2 + d}{\sqrt{4ec - d^2}} \right) \right] d\xi_{\mathbf{k}c}, \quad 4ec > d^2. \end{aligned} \quad (20)$$

In the following section noise-induced effects, which arise during Turing pattern formation in the well-known biophysical system, will be studied. We compare the analytical results obtained using the above-developed approach with the results of numerical experiments.

V. NOISE-INDUCED EFFECTS IN ONE COMPLICATED BIOPHYSICAL SYSTEM

A mathematical model of the system under study is described by equations [38–40]:

$$\begin{aligned} \frac{\partial x_1}{\partial t} &= rx_1(1 - x_1) - \frac{ax_1x_2}{1+bx_1^2} + D_1 \nabla^2 x_1, \\ \frac{\partial x_2}{\partial t} &= \frac{ax_1x_2}{1+bx_1^2} - mx_2 - \frac{g^2 x_2^2}{1+h^2 x_2^2} f + D_2 \nabla^2 x_2, \end{aligned} \quad (21)$$

where x_1, x_2 are state functions, parameters r, a, b, m, g, h, f, D_1 , and D_2 are described in detail in [38, 40]. The investigation of local dynamics and bifurcation analysis of system (21) is carried out in papers [39, 40].

We introduce the dimensionless time $\tau = rt$ and coordinates $\mathbf{x}' = \mathbf{x}\sqrt{r/D_1}$, and represent the parameters m/r and a/r in the form of: $m/r = (m_0/r_0)(1 + f_1(\mathbf{x}', \tau))$, $a/r = (a_0/r_0)(1 + f_2(\mathbf{x}', \tau))$. Here m_0, r_0, a_0 are spatio-temporal averages of the corresponding parameters, random homogeneous isotropic fields $f_i(\mathbf{x}', \tau)$ determine the spatio-temporal Gaussian fluctuations of these parameters and have zero means and correlation functions of the form (3), (10). Taking into account the noise we obtain

$$\begin{aligned}\frac{\partial x_1}{\partial \tau} &= x_1(1 - x_1) - \frac{a_0}{r_0}(1 + f_2(\mathbf{x}', \tau))\frac{x_1 x_2}{1 + b x_1} + D_1 \nabla'^2 x_1, \\ \frac{\partial x_2}{\partial \tau} &= \frac{a_0}{r_0}(1 + f_2(\mathbf{x}', \tau))\frac{x_1 x_2}{1 + b x_1} - \frac{m_0}{r_0}(1 + f_1(\mathbf{x}', \tau))x_2 - \frac{g^2 x_2^2}{1 + h^2 x_2^2}f + D_2 \nabla'^2 x_2,\end{aligned}\quad (22)$$

A. Analytical research

In this section the results of the analytical investigation of system (22) obtained on the basis of equation (15), (16), (18) – (20) are given.

For system (22) a dispersion equation for the unstable mode averaged amplitudes of the form (15) was obtained. Dependencies of real parts of eigenvalue λ of $\langle \xi_{\mathbf{k}u}^{(1)} \rangle$ on the wavenumbers are shown in Fig. 1.

It is evident from the dependencies in Fig.1(a) (dotted and dash-dot lines), that the region of instability of the system (22) with $Re\lambda > 0$ increases with the intensity of external noise θ_2 in the supercritical region. The absolute values of the increments of unstable mode averaged amplitudes are greater in the presence of noise than in its absence. Therefore, unstable mode amplitudes increase, on the average, considerably faster in the presence of external noise than in its absence, which should accelerate the process of spatial pattern formation as compared to the deterministic description. In other words, the destruction of the homogeneous state and pattern formation should take place earlier in time. Besides, the expansion of the instability region and, consequently, increasing the number of interacting unstable modes should result in the changes of the form of the patterns.

Fig. 1(b) shows the dependencies $Re\lambda(k)$ in the subcritical region. It is obvious, that instability does not appear in the absence of noise (solid line in Fig. 1(b)) or in case of its low intensity (dotted line in Fig. 1(b)). However, starting with certain intensity (dash-dot line in Fig. 1(b)) there appears a region with $Re\lambda > 0$, i.e., there exists critical intensity of noise, that induced parametric instability of system (22). Phase transition with pattern formation in the presence of noise should take place in the subcritical region (i.e., earlier by the value of the control parameter than in the case of the deterministic description).

In the subcritical region the state of system is homogeneous and statistically stationary (disorder). Unimodal probability density is known to correspond to this state. The appearance of inhomogeneous statistically stationary state (order) manifests itself in the splitting of probability density maximum into two symmetrical ones.

The Fokker - Planck equation for the critical mode probability density of the form (18) has been derived for the system (22). Figures 2 and 3 show the variation of stationary probability density of this mode with increasing noise intensity when passing through the bifurcation point of a deterministic system.

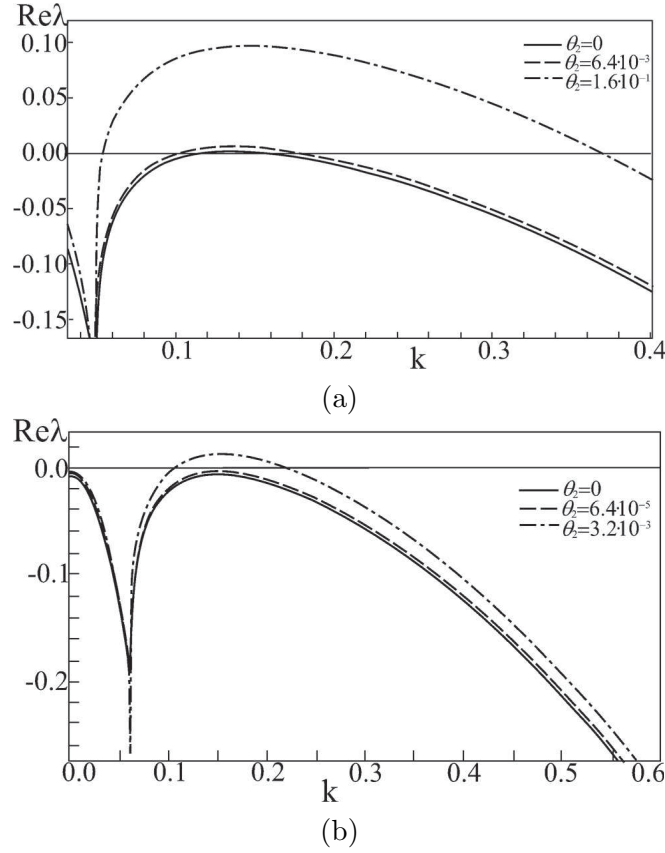


FIG. 1: Real parts of eigenvalue λ of unstable mode averaged amplitudes $\langle \xi_{ku}^{(1)} \rangle$ versus wavenumbers k for system (22) when the noise intensity θ_2 changes. (a) Supercritical region ($D=150$). (b) Subcritical region ($D=100$). For the comparison, the figure shows curves for $\theta_1=\theta_2=0$ (solid line). Other parameters of the model are $r_0=1$, $a_0=8$, $g=1.434$, $f=0.093$, $h=0.857$, $b=11.905$, $m_0=0.490$, $r_{f1}=r_{f2}=1$, $\theta_1=2.4 \cdot 10^{-5}$. The critical control parameter is $D=135$.

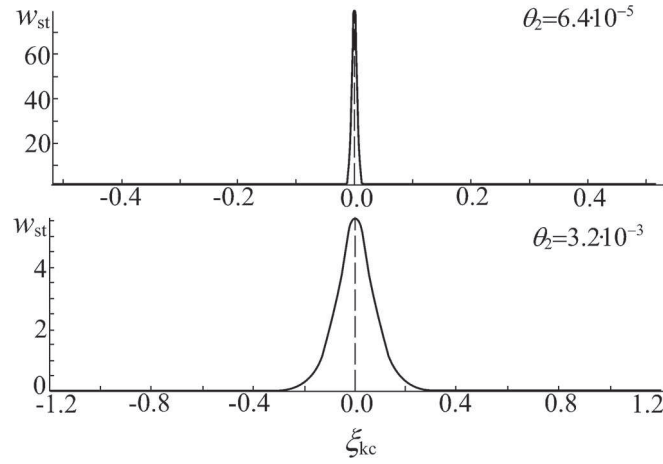


FIG. 2: Steady-state probability density (19) for critical mode values of system (22) in the subcritical region for two values of noise intensity θ_2 . The system parameters are as in Fig.1(a).

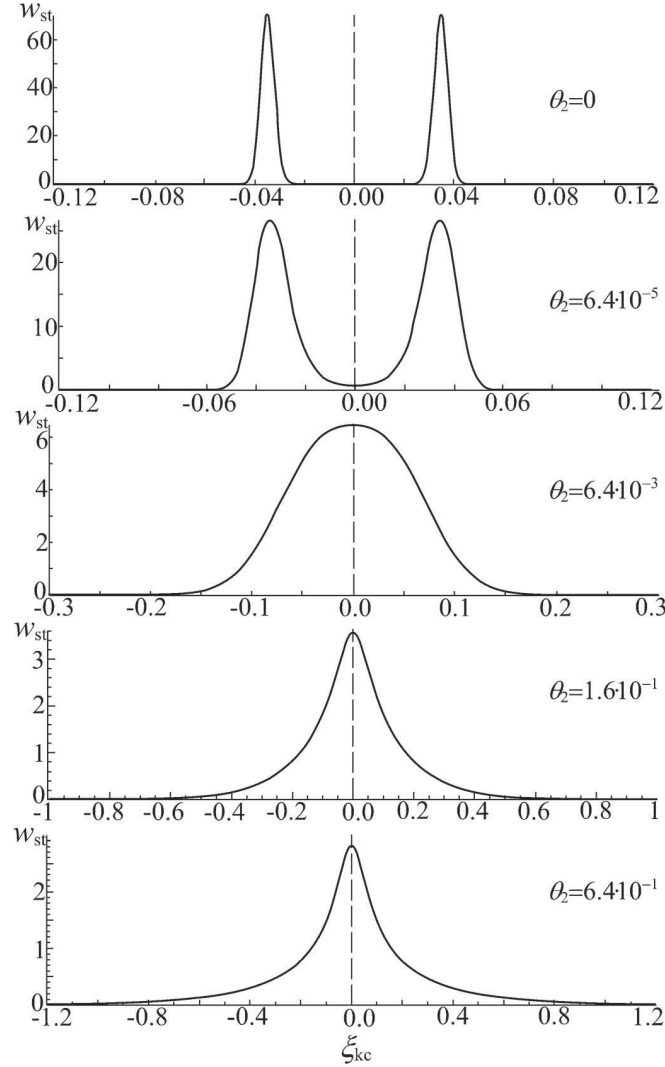


FIG. 3: Steady-state probability density (19) for critical mode values of system (22) in the supercritical region for five values of noise intensity θ_2 . The system parameters are as in Fig.1(b).

Fig. 2 illustrates the probability density of critical modes in the subcritical region. If the noise is low the stationary probability density is close to the δ -function; the average and most probable values of the critical mode coincide and are equal to zero, i.e., the homogeneous statistically stationary state of system is the most probable one (see top Fig.2). Deformation of the curve of stationary probability density occurs if the noise intensity increases (see bottom Fig.2): the maximum is considerably reduced without being shifted; the base of the curve herewith is expanded. As the probability distribution obtained is not Gaussian the mean value becomes different from the most probable one. Thus, although the unimodal probability density is not split into bimodal density the mean value of the order parameter becomes different from zero and we should expect the inhomogeneous statistically stationary state (order) to arise, the probability of which is not great. It is obvious, that the occurrence of a new state is accidental in the case under consideration. It can be explained in the following way. Formation of patterns in the subcritical region may occur on the strong inhomogeneities of medium. Such inhomogeneities may be produced by strong (large-scale)

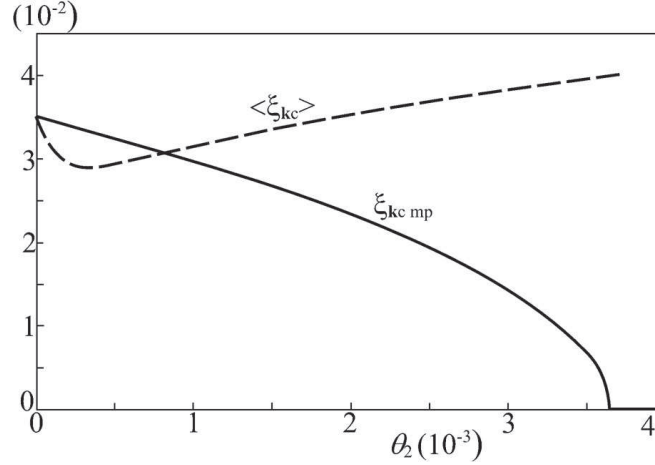


FIG. 4: The steady-state mean $\xi_{kc\ mp}$ and most probable $\xi_{kc\ mp}$ values of the critical mode as a function of noise intensity are given by Eqs. (18)-(20). Supercritical region.

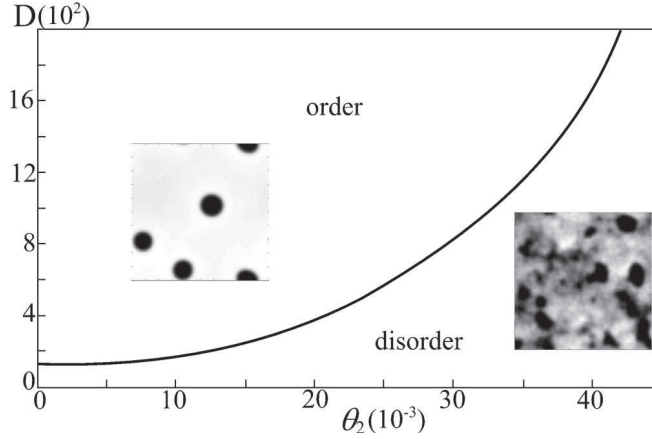


FIG. 5: The boundary of the “order-disorder” transition on the plane of parameters D and θ_2 .

fluctuations, the probability of which is not great and depends on the noise parameters. Thus, given prolonged observation of the system’s evolution and proper parameters of external noise we might expect the formation of such random inhomogeneities, that will cause patterns to occur in the area of their location.

Fig. 3 demonstrates the steady-state probability density of the critical mode (19) of system (22) in the supercritical region at various noise intensity. Bimodal probability densities correspond to the existence of spatial patterns. Herewith, the most probable value and the expectation of the order parameter become different from zero. Fig. 3 clearly shows, that maximums gradually merge as the noise intensity increases, and unimodal density reappears at some critical value of noise intensity. At the same time $\xi_{kc\ mp} = 0$, and $\langle \xi_{kc} \rangle \neq 0$. System (22) transforms to the state of strongly irregular behavior (disorder). Thus, the obtained variation of steady-state probability density of critical mode testifies, that there is a phase transition “disorder - order - disorder” in system (22).

The above-mentioned variation of the statistically stationary mean and most probable values of the critical mode corresponding to the variation of densities shown in Fig. 3 is illustrated in Fig. 4.

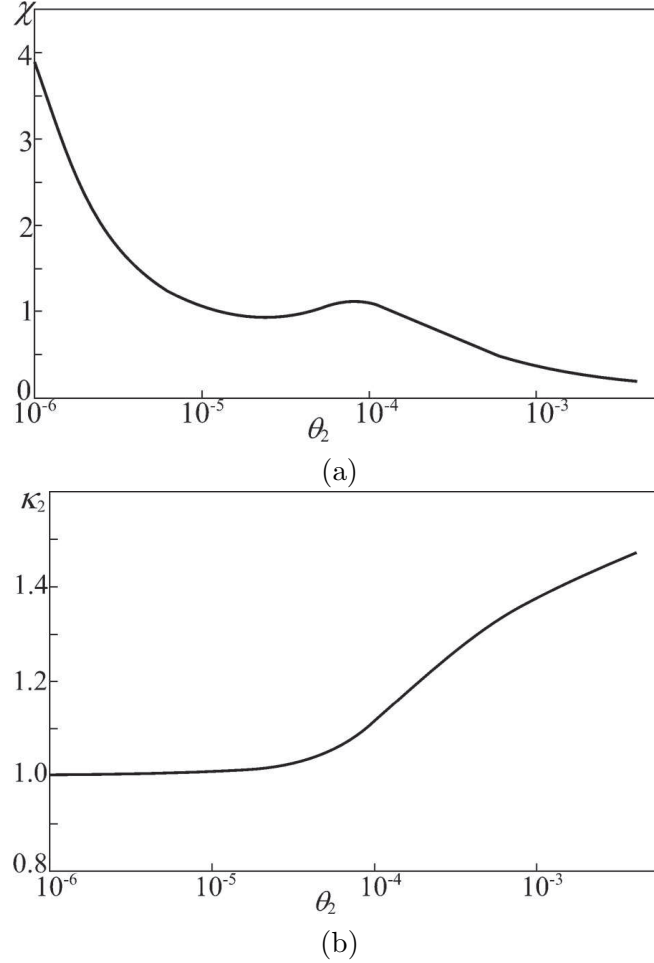


FIG. 6: Susceptibility χ (a) and second-order cumulant κ_2 (b) as functions of noise intensity θ_2 .

Fig. 5 presents the boundary of noise-induced phase transition “order-disorder” for system (22), that is predicted using the approach developed above. It should be noted, that even very small fluctuations will contribute to the loss of stability of the inhomogeneous state and bring about a disordered state when we approaching to the deterministic transition point.

Let us define the relative fluctuations of the order parameter (susceptibility) as $\chi = [\langle \xi_{\mathbf{k}c}^2 \rangle - \langle \xi_{\mathbf{k}c} \rangle^2] / \theta_2$ and its second-order cumulant $\kappa_2 = \langle \xi_{\mathbf{k}c}^2 \rangle / \langle \xi_{\mathbf{k}c} \rangle^2$ in accordance with [12]. In Fig. 6 shows plots of χ and κ_2 as noise intensity functions. The presence of susceptibility maxima (see Fig. 6(a)) clearly shows the increase of fluctuations in the vicinity of two critical points. The form of the curves κ_2 (see Fig. 6(b)) and χ obtained for system (22) coincides qualitatively in the corresponding region with the similar curves obtained numerically in [12] for system, in which a pure noise-induced transition is observed. This testifies, that the noise-induced transition “order-disorder” is of the same nature in system (22).

B. Simulation

This section presents the results of simulation of the evolution of system (22) in two-dimensional space.

The following set of parameters $m_0/r_0 = 0.490$, $a_0/r_0 = 8$, $g^2/r_0 = 2.056$, $f = 0.093$, $h = 0.857$, $b = 11.905$, $D_2/D_1 = 150$, periodic boundary conditions and a rectangular domain of integration are chosen to model the evolution of system (22) in the vicinity of the Turing bifurcation point. Three plane waves with a low amplitude and critical wavenumber propagated at an angle of 60° to each other destabilized the spatial homogeneous state and provided hexagonal symmetry of the initial state. The size of the integration domain is 210.0×186.8 .

The fluctuations of the parameters m/r and a/r are modeled as homogeneous isotropic Gaussian fields with the zero mean and correlation functions of the form (3), (11), where $r_{fi}=1$, $k_{ti}=100$. The values of k_{ti} inverse to the correlation time are chosen so, that all characteristic times of a deterministic problem are considerable greater than the correlation time. The choice of such values of k_{ti} ensures, that the appropriate condition is complied with in theory.

The process of spatial pattern formation in the vicinity of the Turing bifurcation point with increasing noise intensity θ_2 is presented in Fig. 7. In addition to the typical pattern of the state function x_1 distribution over the surface the Fig. 7 also shows the top view giving a more visual representation of the structures configuration. To provide the top view the colour gradient from black to white visualized the variation of values of x_1 from minimum to maximum, respectively. Black areas (“cavities”) correspond to empty regions of space. Fig. 7 shows, that in the presence of noise patterns begin to form earlier than in the deterministic case. Besides, the greater the noise intensity, the earlier the process of formation starts. It should be noted, that noise destroys the symmetry of spatial patterns.

From Fig. 7 we notice, that beginning with certain noise intensity the patterns formed become unstable; alternation of various random pattern configurations takes place; contours of certain “cavities” change, i.e., the system transforms into the state of irregular behavior – disorder.

Fig. 8 demonstrates the evolution of spatial patterns arising spontaneously away from the Turing bifurcation point. The initial conditions correspond to the homogeneous stationary state of the system without initial perturbation. The boundary conditions are periodic. The domain of integration is square 200×200 . The modeling parameters are $m_0/r_0 = 0.490$, $a_0/r_0 = 8$, $g^2/r_0 = 2.056$, $f = 0.093$, $h = 0.857$, $b = 11.905$, $D_2/D_1 = 1000$.

It is obviously from Fig. 8, that all the regularities described above are retained in the evolution of the system away from the bifurcation point. It must be emphasized here, that the “order-disorder” transition in this case takes place at greater values of noise intensity than in the vicinity of the bifurcation point.

Fig. 9 gives a clear idea of the system’s behavior at high level of noise. As opposed to pure noise-induced transition here the system’s random behavior is conditioned by fast formation and destruction of a great number of patterns having random contours and arising in random places.

In the light of the theory developed in our study such behavior is due to the fact, that the region of the system’s instability considerable expands at high level of noise and pattern formation takes place owing to cooperation and competition of a very large number of unstable modes with different conditions of resonant interaction.

We have also carried out simulation of system (22) evolution in the subcritical region. The following parameters $m_0/r_0 = \mu_1 = 0.510$, $a_0/r_0 = \mu_2 = 8$, $g^2/r_0 = 2.06$, $f = 0.093$, $h = 0.857$, $b = 11.905$, $D_2/D_1 = 1000$, $\theta_1=2.6 \cdot 10^{-3}$, $\theta_2=2.56 \cdot 10^{-2}$; $k_{f1} = k_{f2} = 0.5$, the domain 200×200 , and periodic boundary conditions are chosen for the modeling.

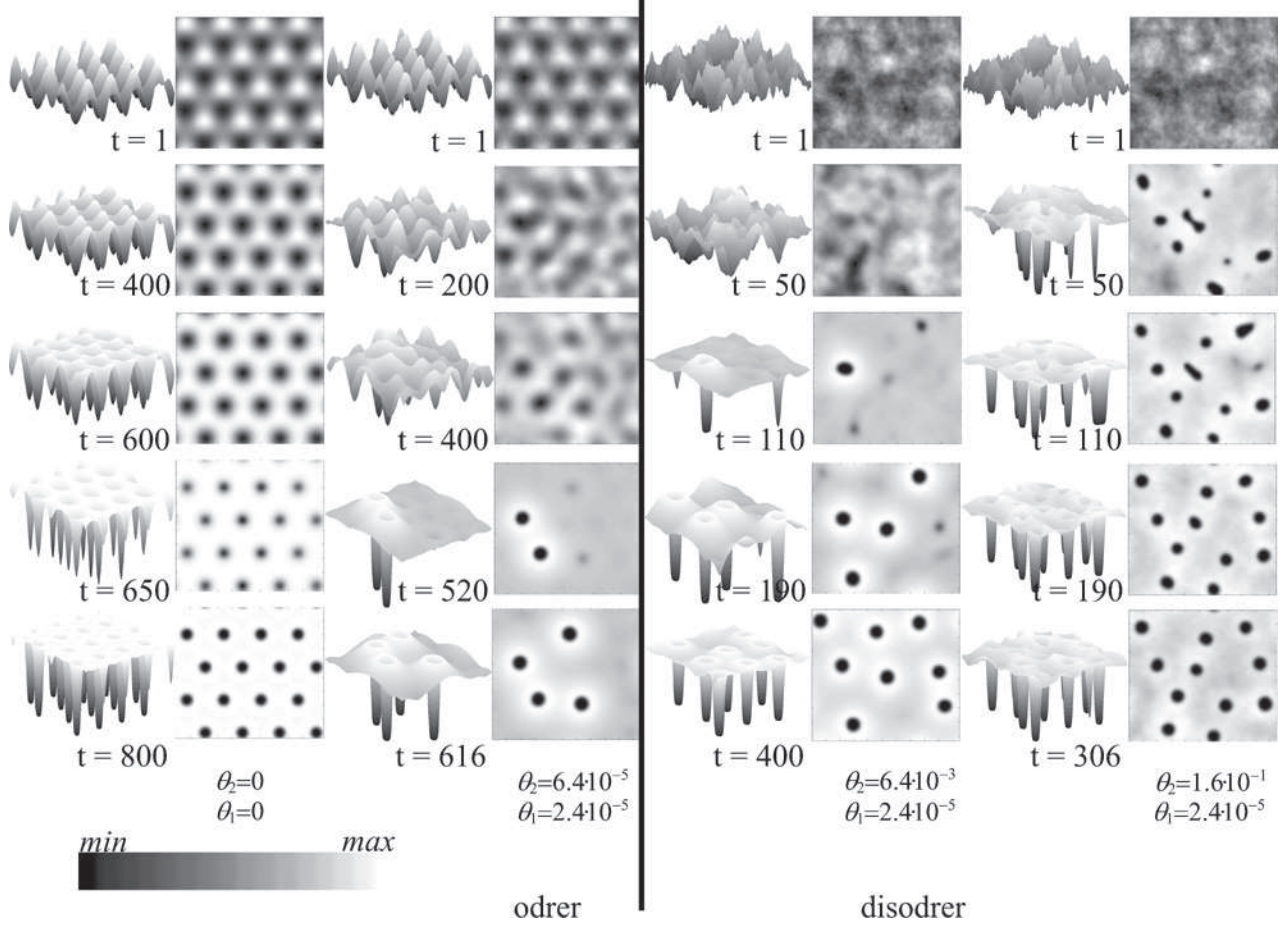


FIG. 7: The evolution of spatial Turing patterns at different noise intensities near the deterministic transition point. On the figure the moments of computer time corresponding to a given distribution of the dynamic variable x_1 of system (22) are indicated. The bottom row of images shows statistically steady state in the case of order.

Fig. 10 presents the evolution of x_1 state function distribution for the instants of time $t = 1, 927, 940, 950$ and 1120 .

The pattern arising under parametric instability has a “solitonlike” shape (see Fig. 10 $t=1120$). From Fig. 10 it is also clear, that the duration of the process of destroying the homogeneous state in the subcritical region (see Fig. 10 $t=1-927$) is considerably longer than that in the supercritical region (see Fig. 8 $t=1-40$ in the case of order). As predicted in section V.A, the initiation of inhomogeneous state in the subcritical region may be seen only as a result of prolonged observation of the system’s evolution.

Thus, the results of simulation of system (22) evolution qualitatively fully confirm the theoretical conclusions of section V. A comparison of Fig. 5 and Fig. 7 makes it possible to establish a satisfactory quantitative correspondence between the theory and the numerical experiment in the vicinity of the Turing bifurcation point.

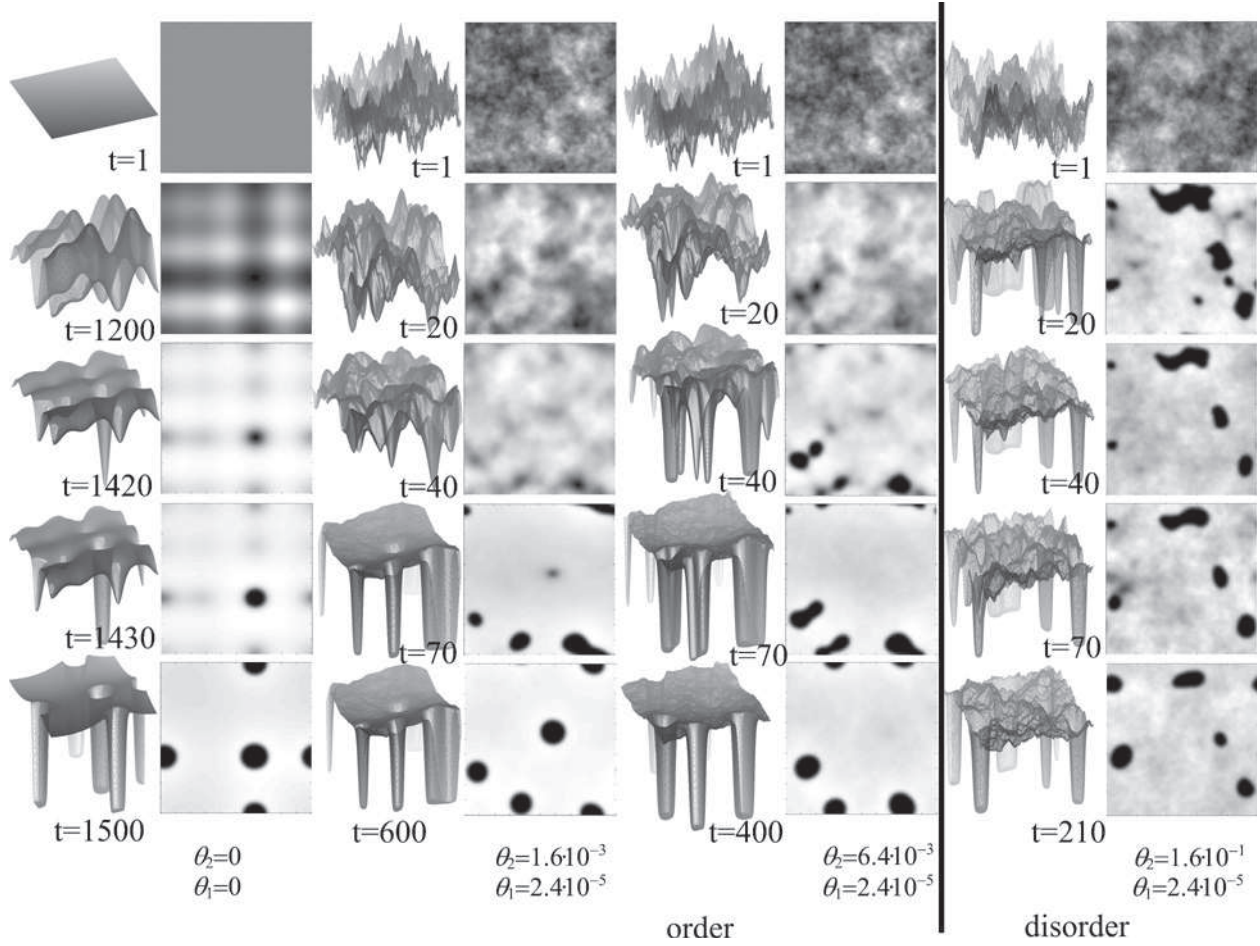


FIG. 8: The evolution of spatial Turing patterns at different noise intensities away from the deterministic transition point. See explanation for Fig. 7.

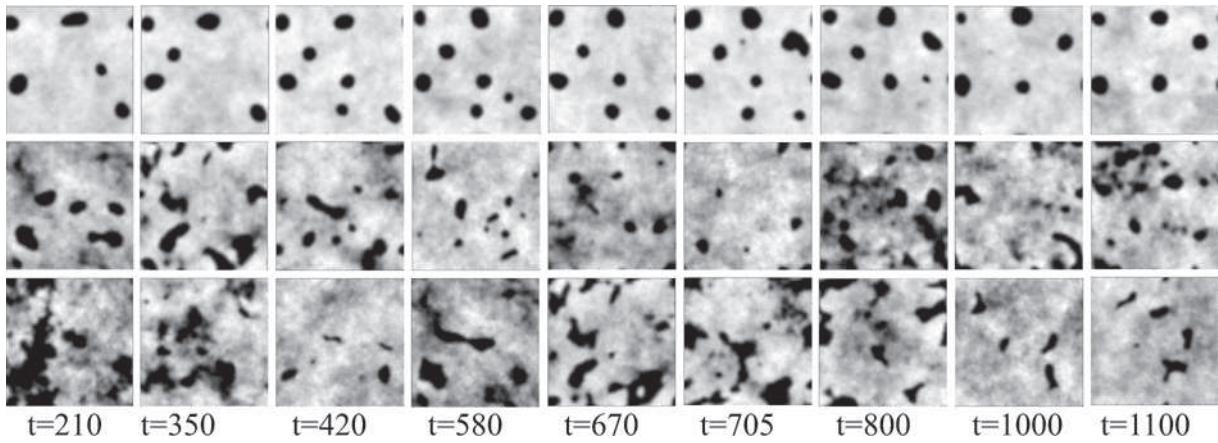


FIG. 9: The evolution of spatial Turing patterns in a strong noise. The figure shows a strongly irregular behavior (disorder) of the system (22). The top horizontal row: $\theta_2=0.16$. The center horizontal row: $\theta_2=0.64$. The bottom horizontal row: $\theta_2=1.44$, $\theta_1=2.40 \cdot 10^{-5}$, $r_{f1} = r_{f2}=1$.

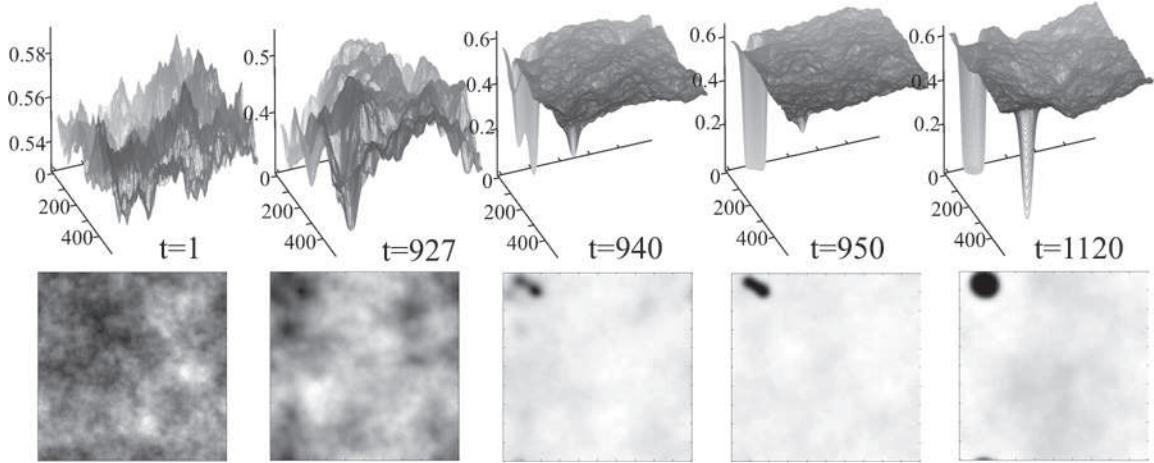


FIG. 10: Noise-induced parametric excitation of spatial Turing pattern in the subcritical region.

VI. CONCLUSION

In our study a theory is developed, that makes it possible to predict and analyze in detail different noise-induced effects occurring in open nonlinear distributed multicomponent multidimensional systems both in the vicinity of and away from the deterministic transition point. The prediction and analysis are based on the unified point of view of the concept of order parameters.

Stochastic equations have been obtained for amplitudes of unstable modes (order parameters), as well as dispersion equation for averaged amplitudes of unstable modes. The dependence of eigenvalues of unstable mode averaged amplitudes on wavenumbers, noise intensity, and correlation length has been found analytically. The Fokker-Planck equation for the order parameters of the systems under study has been obtained. Its stationary solution for a critical mode has been obtained in an explicit form.

As our theory predicts, the increments of unstable modes change, the system's instability region is extended, the conditions of mode resonance interaction change, and noise-induced parametric instability occurs in external noise. The destruction of a homogeneous state and the pattern formation take place faster than in the deterministic case. Our theory points to the existence of a noise-induced "disorder - order - disorder" phase transition in systems of the type discussed.

The advantages of the approach developed lies in the fact, that it applies to multicomponent multidimensional systems. The stationary solution to the Fokker-Planck equation for the critical order parameter is written in an explicit form. The approach applies to a wider class of functions $P_{1j}(x_1, x_2, \chi_{m+1}, \dots, \chi_{n'})$, $P_{2j}(x_1, x_2, \eta_{l+1}, \dots, \eta_{s'})$ including those with a discontinuity of the second kind. Moreover, the approach suggested does not contain an arbitrary element connected with the discretization of system's continuous space. It can also be used in the case of noise with finite characteristic spatial and temporal scales.

The numerical analysis of a specific system of the type considered qualitatively confirms the theoretical conclusions and provides a satisfactory quantitative correspondence between the theory and the numerical experiment in the vicinity of the bifurcation point.

Appendix A

Functions $\sigma_{j'j''}^{(j)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'')$, $\sigma_{j'j''j'''}^{(j)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'', \mathbf{k}''')$, $\varepsilon_{\varphi,j'}^{(j)}(\mathbf{k}, \mathbf{k}')$, $\varepsilon_{\varphi,j'j''}^{(j)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'')$ introduced in equations (10):

$$\sigma_{j'j''}^{(j)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'') = \sum_{\varepsilon, \mu, \nu} g_{\varepsilon, \mu \nu}^{(2)} O_{\varepsilon}^{*(j)}(\mathbf{k}) O_{\mu}^{(j')}(\mathbf{k}') O_{\nu}^{(j'')}(\mathbf{k}''),$$

$$\sigma_{j'j''j'''}^{(j)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'', \mathbf{k}''') = \sum_{\varepsilon, \mu, \nu, \kappa} g_{\varepsilon, \mu \nu \kappa}^{(3)} O_{\varepsilon}^{*(j)}(\mathbf{k}) O_{\mu}^{(j')}(\mathbf{k}') O_{\nu}^{(j'')}(\mathbf{k}'') O_{\kappa}^{(j''')}(\mathbf{k}'''),$$

$$\varepsilon_{\varphi,j'}^{(j)}(\mathbf{k}, \mathbf{k}') = \sum_{\mu} p_{\varphi, \mu}^{(1)} O_{\varphi}^{*(j)}(\mathbf{k}) O_{\mu}^{(j')}(\mathbf{k}'),$$

$$\varepsilon_{\varphi,j'j''}^{(j)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'') = \sum_{\mu, \nu} p_{\varphi, \mu \nu}^{(2)} O_{\varphi}^{*(j)}(\mathbf{k}) O_{\mu}^{(j')}(\mathbf{k}') O_{\nu}^{(j'')}(\mathbf{k}'').$$

Appendix B

Function $\omega(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}''_u, \mathbf{k}'''_u, \mathbf{k}_u - \mathbf{k}'_u)$ and others introduced in equations (14):

$$\begin{aligned} \omega(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}''_u, \mathbf{k}'''_u, \mathbf{k}_u - \mathbf{k}'_u) &= \sigma_{111}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}''_u, \mathbf{k}'''_u) - \\ &- \frac{[\sigma_{11}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}_s) + \sigma_{11}^{(1)}(\mathbf{k}_u, \mathbf{k}_s, \mathbf{k}'_u)]}{\lambda_1(\mathbf{k}_s)} \sigma_{11}^{(1)}(\mathbf{k}_s, \mathbf{k}''_u, \mathbf{k}'''_u) - \\ &- \frac{[\sigma_{12}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}_s) + \sigma_{21}^{(1)}(\mathbf{k}_u, \mathbf{k}_s, \mathbf{k}'_u)]}{\lambda_2(\mathbf{k}_s)} \sigma_{11}^{(2)}(\mathbf{k}_s, \mathbf{k}''_u, \mathbf{k}'''_u), \end{aligned}$$

$$\zeta_{\varphi, \psi, \varphi'}(\mathbf{k}_u, \mathbf{k}_s) = \varepsilon_{\varphi, \psi}^{(1)}(\mathbf{k}_u, \mathbf{k}_s) \frac{O_{\varphi'}^{*(\psi)}(\mathbf{k}_s)}{\lambda_{\psi}(\mathbf{k}_s)} p_{\varphi'}^{(0)},$$

$$\begin{aligned} \eta_{\varphi}(\mathbf{k}_u, \mathbf{k}'_u) &= \varepsilon_{\varphi, 1}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u) - \\ &- \sum_{\psi=1}^2 [\sigma_{1\psi}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}_u - \mathbf{k}'_u) + \sigma_{\psi 1}^{(1)}(\mathbf{k}_u, \mathbf{k}_u - \mathbf{k}'_u, \mathbf{k}'_u)] \frac{O_{\varphi}^{*(\psi)}(\mathbf{k}_u - \mathbf{k}'_u)}{\lambda_{\psi}(\mathbf{k}_u - \mathbf{k}'_u)} p_{\varphi}^{(0)}, \\ \nu_{\varphi}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}''_u) &= \varepsilon_{\varphi, 11}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}''_u) - \\ &- \sum_{\psi=1}^2 [\sigma_{1\psi}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}_u - \mathbf{k}'_u) + \sigma_{\psi 1}^{(1)}(\mathbf{k}_u, \mathbf{k}_u - \mathbf{k}'_u, \mathbf{k}'_u)] \frac{\varepsilon_{\varphi, 1}^{(\psi)}(\mathbf{k}_u - \mathbf{k}'_u, \mathbf{k}''_u)}{\lambda_{\psi}(\mathbf{k}_u - \mathbf{k}'_u)} - \\ &- \sum_{\psi, \varphi=1}^2 \varepsilon_{\varphi, \psi}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u + \mathbf{k}''_u) \frac{\sigma_{11}^{(\psi)}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}''_u)}{\lambda_{\psi}(\mathbf{k}'_u + \mathbf{k}''_u)}, \end{aligned}$$

$$\begin{aligned}
A_{\varphi,\psi,\varphi'}(\mathbf{k}_u, \mathbf{k}_s, \mathbf{k}'_u) &= \varepsilon_{\varphi,\psi}^{(1)}(\mathbf{k}_u, \mathbf{k}_s) \frac{\varepsilon_{\varphi',1}^{(\psi)}(\mathbf{k}_s, \mathbf{k}'_u)}{\lambda_{\psi}(\mathbf{k}_s)}, \\
B_{\varphi,\psi,\varphi'}(\mathbf{k}_u, \mathbf{k}_s, \mathbf{k}'_u) &= [\varepsilon_{\varphi,1\psi}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}_s) + \varepsilon_{\varphi,\psi 1}^{(1)}(\mathbf{k}_u, \mathbf{k}_s, \mathbf{k}'_u)] \frac{O_{\varphi'}^{*(\psi)}(\mathbf{k}_s)}{\lambda_{\psi}(\mathbf{k}_s)} p_{\varphi'}^{(0)}, \\
C_{\varphi,\psi,\varphi'}(\mathbf{k}_u, \mathbf{k}_s, \mathbf{k}'_u, \mathbf{k}''_u) &= \varepsilon_{\varphi,\psi}^{(1)}(\mathbf{k}_u, \mathbf{k}_s) \frac{\varepsilon_{\varphi',11}^{(\psi)}(\mathbf{k}_s, \mathbf{k}'_u, \mathbf{k}''_u)}{\lambda_{\psi}(\mathbf{k}_s)}, \\
D_{\varphi,\psi,\varphi'}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}_s, \mathbf{k}''_u) &= [\varepsilon_{\varphi,1\psi}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}_s) + \varepsilon_{\varphi,\psi 1}^{(1)}(\mathbf{k}_u, \mathbf{k}_s, \mathbf{k}'_u)] \frac{\varepsilon_{\varphi',1}^{(\psi)}(\mathbf{k}_s, \mathbf{k}''_u)}{\lambda_{\psi}(\mathbf{k}_s)}, \\
E_{\psi,\varphi}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}_u - \mathbf{k}'_u, \mathbf{k}''_u, \mathbf{k}'''_u) &= \\
&= [\sigma_{1\psi}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}_u - \mathbf{k}'_u) + \sigma_{\psi 1}^{(1)}(\mathbf{k}_u, \mathbf{k}_u - \mathbf{k}'_u, \mathbf{k}'_u)] \frac{\varepsilon_{\varphi,11}^{(\psi)}(\mathbf{k}_u - \mathbf{k}'_u, \mathbf{k}''_u, \mathbf{k}'''_u)}{\lambda_{\psi}(\mathbf{k}_u - \mathbf{k}'_u)}, \\
F_{\psi,\varphi}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}''_u, \mathbf{k}'''_u) &= \\
&= [\varepsilon_{\varphi,1\psi}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}''_u + \mathbf{k}'''_u) + \varepsilon_{\varphi,\psi 1}^{(1)}(\mathbf{k}_u, \mathbf{k}''_u + \mathbf{k}'''_u, \mathbf{k}'_u)] \frac{\sigma_{11}^{(\psi)}(\mathbf{k}''_u + \mathbf{k}'''_u, \mathbf{k}''_u, \mathbf{k}'''_u)}{\lambda_{\psi}(\mathbf{k}''_u + \mathbf{k}'''_u)}, \\
G_{\psi,\varphi,\varphi'}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}_s, \mathbf{k}''_u, \mathbf{k}'''_u) &= \\
&= [\varepsilon_{\varphi,1\psi}^{(1)}(\mathbf{k}_u, \mathbf{k}'_u, \mathbf{k}_s) + \varepsilon_{\varphi,\psi 1}^{(1)}(\mathbf{k}_u, \mathbf{k}_s, \mathbf{k}'_u)] \frac{\varepsilon_{\varphi',11}^{(\psi)}(\mathbf{k}_s, \mathbf{k}''_u, \mathbf{k}'''_u)}{\lambda_{\psi}(\mathbf{k}_s)}.
\end{aligned}$$

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